Arithmetic and Geometric Series: Review

Fact — We can write sequences in different ways:

1. $x_n = g(n)$ - using a formula
2. $x_{n+1} = f(x_n), x_1 = a$ - using a rule based on previous terms.

Tip

Don't forget the first term if using a recursive formula!

Example

Write down using both forms, formulae for:

- $1, 5, 9, 13, \ldots$
- $12, 6, 3, \frac{3}{2}, \ldots$

Fact — For an arithemtic sequence, with first term a and common difference d, the sum of the first n terms is:

$$S_n = \underbrace{n}_{\text{number of terms}} \left(\underbrace{\frac{a + a + (n-1)d}{2}}_{\text{average of first and last terms}} \right) = \frac{n}{2}(2a + (n-1)d)$$

For a geometric sequence, with first term a and common ratio r, the sum of the first n terms is:

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

If $|r| < 1$, then

$$S_\infty = \frac{a}{1 - r}$$

Example

A student is reading a 426-page book finds that he reads faster as he gets into the subject. He reads 19 pages on the first day, and his rate of reading then goes up by 3 pages each day. How long does he take to finish the book?

Tip

If a, b, c are in arithmetic progression, then:

$$b - a = c - b \Leftrightarrow b = \frac{a + c}{2}$$

If a, b, c are in geometric progression, then:

$$\frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$$

Solving equations numerically: Review

Fact — If the sequence given by the recursive definition $x_{r+1} = F(x_r)$, with some initial value *converges to a limit* l, then l is a root of the equation $x = F(x)$

Tip

A sequence given be a recursive definition $x_{r+1} = F(x_r)$ doesn't necessarily converge!

Example

Suppose we want to solve the equation $x^3 - 3x - 5 = 0$. We can write this as:

- $x = \frac{1}{3}(x^3 - 5)$, ie $x_{r+1} = \frac{1}{3}(x_r^3 - 5)$
- $x = \sqrt[3]{3x + 5}$, ie $x_{r+1} = \sqrt[3]{3x_r + 5}$

Starting from $x_0 = 2$, write down first 5 terms of each iteration. Find a root to 3 (s.f.).

Fact (Convergence of Iterative Methods) — If the iteration $x_{r+1} = F(x_r)$ is used to find approximations of a root α, then the sequence of errors is approximately *geometric* with common ratio $F'(\alpha)$ (assuming $F'(\alpha) \neq 0$)

Fact (Convergence of Iterative Methods) — If the method has $F'(\alpha) = 0$ then the convergence is *quadratic* (or better).

Fact (Newton-Raphson) — To find a root of $f(x) = 0$, consider using the iterative method:

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

ie with $F(x) = x - \dfrac{f(x)}{f'(x)}$

Example

Suppose $f(x) = x^3 - 3x - 5$, use Newton-Raphson to find a **root** near 2 to 6 (d.p.)

Standard Series

Fact —

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Tip

You need to be able to prove these results! (Induction!)

Example

Prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ by induction

Fact —

$$\begin{aligned}\sum (a_n + b_n) &= \sum a_n + \sum b_n \\ \sum c a_n &= c \sum a_n\end{aligned}$$

Example

Find a formula for $1 \times 2 \times 4 + 2 \times 3 \times 5 + \cdots + n(n+1)(n+3)$.

Method of Differences - Telescoping Series

Example

Find the sum $\sum_{k=1}^{n} \frac{1}{k^2 + k}$

Fact (Method of Differences) — If $g(r) = f(r+1) - f(r)$

$$\sum_{r=1}^{n} g(r) = \sum_{r=1}^{n} (f(r+1) - f(r)) = f(n+1) - f(1)$$

Example

Expand $(r+1)^3 - r^3$, and use your result to derive a formula for $\sum_{r=1}^{n} r^2$.

Proof by Induction

Example

Prove that $3 \times 7^{2n} + 1$ is divisible by 4 for $n \in \mathbb{N}$

Example

Prove that $2^n > 2n$ for $n \geq 3$

Example

Use induction to prove that the inequality $2^n > 6n + 1$ holds for all integers $n \geq 5$.

Example

Suppose $x_{n+1} = 3x_n - 1, x_1 = 2$, then $x_n = \frac{1}{2}(3^n + 1)$

Example

A sequence of integers is defined recursively by the relation

$$a_{n+1} = a_n - 4, a_1 = 3, n = 1, 2, 3, \ldots$$

Prove by induction that its n^{th} term is given by

$$a_n = 7 - 4n$$